

Ex 1:

$$I_m = \int_0^1 t^m \sqrt{1+t} dt \quad ; \quad J_m = n I_m. \quad (1)$$

$$1- \quad I_0 = \int_0^1 \sqrt{1+t} dt = \left[\frac{2}{3} (1+t)^{3/2} \right]_0^1 = \frac{2}{3} (2\sqrt{2} - 1)$$

$$2- \quad I_1 = \int_0^1 t \sqrt{1+t} dt \quad \begin{array}{l} u = t \Rightarrow u' = 1 \\ v' = \sqrt{1+t} \Rightarrow v = \frac{2}{3} (1+t)^{3/2} \end{array}$$
$$= \left[\frac{2}{3} t (1+t)^{3/2} \right]_0^1 - \frac{2}{3} \int_0^1 (1+t)^{3/2} dt$$
$$= \frac{4}{3} \sqrt{2} - \frac{2}{3} \times \frac{2}{5} \left[(1+t)^{5/2} \right]_0^1 = \frac{4}{3} \sqrt{2} - \frac{4}{15} (4\sqrt{2} - 1) = \frac{4}{15} (\sqrt{2} + 1)$$

$$3- \quad I_{m+1} - I_m = \int_0^1 t^{m+1} \sqrt{1+t} dt - \int_0^1 t^m \sqrt{1+t} dt$$
$$= \int_0^1 \underbrace{(t-1)}_{\leq 0} \underbrace{t^m \sqrt{1+t}}_{\geq 0} dt \leq 0 \Rightarrow (I_m) \searrow$$

$$4- \quad 1 \leq \sqrt{1+t} \leq \sqrt{2} \quad \text{Pour } t \in [0, 1]$$

$$\Rightarrow \int_0^1 t^m dt \leq I_m \leq \int_0^1 \sqrt{2} t^m dt$$

$$\Rightarrow \left[\frac{t^{m+1}}{m+1} \right]_0^1 \leq I_m \leq \sqrt{2} \left[\frac{t^{m+1}}{m+1} \right]_0^1$$

$$\Rightarrow \frac{1}{m+1} \leq I_m \leq \frac{\sqrt{2}}{m+1}$$

$$5- \quad \lim_{m \rightarrow \infty} I_m = 0$$

$$6- \quad t \leq 1 \Rightarrow 1+t \leq 2 \Rightarrow \sqrt{1+t} \leq \sqrt{2}$$

$$\Rightarrow 0 \leq \sqrt{2} - \sqrt{1+t}$$

$$\sqrt{2} - \sqrt{1+t} = \frac{(\sqrt{2} - \sqrt{1+t})(\sqrt{2} + \sqrt{1+t})}{\sqrt{2} + \sqrt{1+t}} = \frac{2-t-1-t}{\sqrt{2} + \sqrt{1+t}} = \frac{1-t}{\sqrt{2} + \sqrt{1+t}}$$

$$\text{Or } \sqrt{2} \geq 1 \Rightarrow \sqrt{2} + 1 \geq 2 \Rightarrow \sqrt{2} + \sqrt{1+t} \geq 2 \text{ pour } t \in [0,1]$$

$$\Rightarrow \frac{1}{\sqrt{2} + \sqrt{1+t}} \leq \frac{1}{2} \cdot \text{Donc } \sqrt{2} - \sqrt{1+t} \leq \frac{1-t}{2}$$

$$7- \quad \sqrt{2} - \sqrt{1+t} \leq \frac{1-t}{2} \Rightarrow \sqrt{2} + \frac{t-1}{2} \leq \sqrt{1+t} \quad \forall t \in [0,1]$$

$$\Rightarrow \int_0^1 \left(\sqrt{2} + \frac{t-1}{2} \right)^m dt \leq I_m \Rightarrow \left[\frac{\sqrt{2} t^{m+1}}{m+1} + \frac{t^{m+2}}{2(m+2)} - \frac{t^{m+1}}{2(m+1)} \right]_0^1 \leq I_m$$

$$\Rightarrow \frac{\sqrt{2}}{m+1} + \frac{1}{2} \left(\frac{1}{m+2} - \frac{1}{m+1} \right) \leq I_m \Rightarrow \frac{\sqrt{2}}{m+1} + \frac{1}{2} \frac{m+1-m-2}{(m+2)(m+1)} \leq I_m$$

$$\Rightarrow \frac{\sqrt{2}}{m+1} + \frac{1}{2} \frac{-1}{(m+2)(m+1)} \leq I_m \Rightarrow \frac{\sqrt{2}}{m+1} - \frac{1}{2m^2} \leq I_m$$

$$\text{Or } I_m \stackrel{(6)}{\leq} \frac{\sqrt{2}}{m+1} \Rightarrow \frac{\sqrt{2}}{m+1} - \frac{1}{2m^2} \leq I_m \leq \frac{\sqrt{2}}{m+1}$$

$$8- \text{ On a donc } \underbrace{\frac{\sqrt{2} \frac{m}{m+1} - \frac{1}{2m}}{\searrow \sqrt{2}}} \leq \underbrace{I_m}_{= I_m} \leq \underbrace{\frac{\sqrt{2} \frac{m}{m+1}}{\searrow \sqrt{2}}}$$

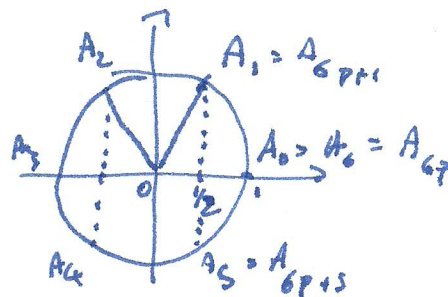
$$\text{donc } \lim_{m \rightarrow \infty} I_m = \sqrt{2}$$

Ex 2:

$$z_1 = \frac{1}{2} + i \frac{\sqrt{3}}{2}$$

(3)

1. $|z_1| = \frac{1}{4} + \frac{3}{4} = 1$ donc



2. a) $z_2 = z_1^2 = \frac{1}{4} - \frac{3}{4} + \frac{2i\sqrt{3}}{4} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

b) $|z_m| = |z_1^m| = |z_1|^m = 1^m = 1$

$\Rightarrow A_m \in \overline{\mathcal{C}}(0; 1)$

3- $z_{m+1} - z_m = z_1^{m+1} - z_1^m = z_1^m (z_1 - 1) = z_1^m \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

4- $|z_{m+1} - z_m| = |z_1|^m \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = 1 \times 1 = 1 = A_m A_{m+1}$

5- $\arg A_m = \arg A_{m+1} = A_m A_{m+1} = 1 \Rightarrow \arg A_m A_{m+1}$ opposés

$\Rightarrow \widehat{A_m A_{m+1}} = \frac{\pi}{3}$

6. $6 \times \frac{\pi}{3} = 2\pi \Rightarrow A_0 = A_6$

