Ex 1 : $I_n = \int_0^1 t^n \sqrt{1+t}\, dt$; $J_n = n I_n$.

1- $I_0 = \int_0^1 \sqrt{1+t}\, dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^1 = \frac{2}{3}\left(2\sqrt{2} - 1\right)$

2- $I_1 = \int_0^1 t\sqrt{1+t}\, dt$ $\quad u = t \Rightarrow u' = 1$

$v' = \sqrt{1+t} \Rightarrow v = \frac{2}{3}(1+t)^{3/2}$

$= \left[\frac{2}{3} t (1+t)^{3/2}\right]_0^1 - \frac{2}{3}\int_0^1 (1+t)^{3/2} dt$

$= \frac{4}{3}\sqrt{2} - \frac{2}{3} \times \frac{2}{5}\left[(1+t)^{5/2}\right]_0^1 = \frac{4}{3}\sqrt{2} - \frac{4}{15}\left(4\sqrt{2} - 1\right) = \frac{4}{15}\left(\sqrt{2}+1\right)$

3- $I_{n+1} - I_n = \int_0^1 t^{n+1}\sqrt{1+t}\, dt - \int_0^1 t^n \sqrt{1+t}\, dt$

$= \int_0^1 \underbrace{(t-1)}_{\leq 0}\, \underbrace{t^n \sqrt{1+t}}_{\geq 0}\, dt \leq 0 \Rightarrow (I_n) \searrow$

4- $1 \leq \sqrt{1+t} \leq \sqrt{2}$ Pour $t \in [0,1]$

$\Rightarrow \int_0^1 t^n dt \leq I_n \leq \int_0^1 \sqrt{2}\, t^n dt$

$\Rightarrow \left[\frac{t^{n+1}}{n+1}\right]_0^1 \leq I_n \leq \sqrt{2}\left[\frac{t^{n+1}}{n+1}\right]_0^1$

$\Rightarrow \frac{1}{n+1} \leq I_n \leq \frac{\sqrt{2}}{n+1}$

5- $\lim_{n \to \infty} I_n = 0$.

6- $t \leq 1 \Rightarrow 1+t \leq 2 \Rightarrow \sqrt{1+t} \leq \sqrt{2}$

$\Rightarrow 0 \leq \sqrt{2} - \sqrt{1+t}$.

$$\sqrt{2} - \sqrt{1+t} = \frac{(\sqrt{2} - \sqrt{1+t})(\sqrt{2} + \sqrt{1+t})}{\sqrt{2} + \sqrt{1+t}} = \frac{2-1-t}{\sqrt{2}+\sqrt{1+t}} = \frac{1-t}{\sqrt{2}+\sqrt{1+t}}$$

Or $\sqrt{2} \geq 1 \Rightarrow \sqrt{2}+1 \geq 2 \Rightarrow \sqrt{2} + \sqrt{1+t} \geq 2$ pour $t \in [0,1]$

$\Rightarrow \dfrac{1}{\sqrt{2}+\sqrt{1+t}} \leq \dfrac{1}{2}$. Donc $\sqrt{2} - \sqrt{1+t} \leq \dfrac{1-t}{2}$.

7- $\sqrt{2} - \sqrt{1+t} \leq \dfrac{1-t}{2} \Rightarrow \sqrt{2} + \dfrac{t-1}{2} \leq \sqrt{1+t} \quad \forall t \in [0,1]$

$$\Rightarrow \int_0^1 \left(\sqrt{2} + \frac{t-1}{2}\right) t^n \, dt \leq I_n \Rightarrow \left[\frac{\sqrt{2}\, t^{n+1}}{n+1} + \frac{t^{n+2}}{2(n+2)} - \frac{t^{n+1}}{2(n+1)}\right]_0^1 \leq I_n$$

$$\Rightarrow \frac{\sqrt{2}}{n+1} + \frac{1}{2}\left(\frac{1}{n+2} - \frac{1}{n+1}\right) \leq I_n \Rightarrow \frac{\sqrt{2}}{n+1} + \frac{1}{2}\,\frac{n+1-n-2}{(n+2)(n+1)} \leq I_n$$

$$\Rightarrow \frac{\sqrt{2}}{n+1} + \frac{1}{2}\,\frac{-1}{(n+2)(n+1)} \leq I_n \Rightarrow \frac{\sqrt{2}}{n+1} - \frac{1}{2n^2} \leq I_n$$

Or $I_n \overset{(4)}{\leq} \dfrac{\sqrt{2}}{n+1} \Rightarrow \dfrac{\sqrt{2}}{n+1} - \dfrac{1}{2n^2} \leq I_n \leq \dfrac{\sqrt{2}}{n+1}$.

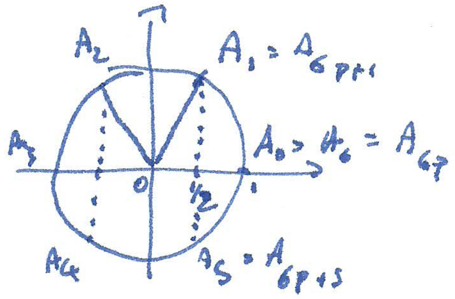
8- On a donc

$$\underbrace{\sqrt{2}\,\frac{n}{n+1} - \frac{1}{2n}}_{\to \sqrt{2}} \leq \underbrace{n I_n}_{= J_n} \leq \underbrace{\sqrt{2}\,\frac{n}{n+1}}_{\to \sqrt{2}}$$

donc $\lim\limits_{n \to \infty} J_n = \sqrt{2}$.

Ex 2 :

$$z_1 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$$



1- $|z_1| = \frac{1}{4} + \frac{3}{4} = 1$ donc

2- a) $z_2 = z_1^2 = \frac{1}{4} - \frac{3}{4} + \frac{2i\sqrt{3}}{4} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

b) $|z_n| = |z_1^n| = |z_1|^n = 1^n = 1$

$\Rightarrow A_n \in \mathcal{C}(O\,;1)$

3- $z_{n+1} - z_n = z_1^{n+1} - z_1^n = z_1^n (z_1 - 1) = z_1^n \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

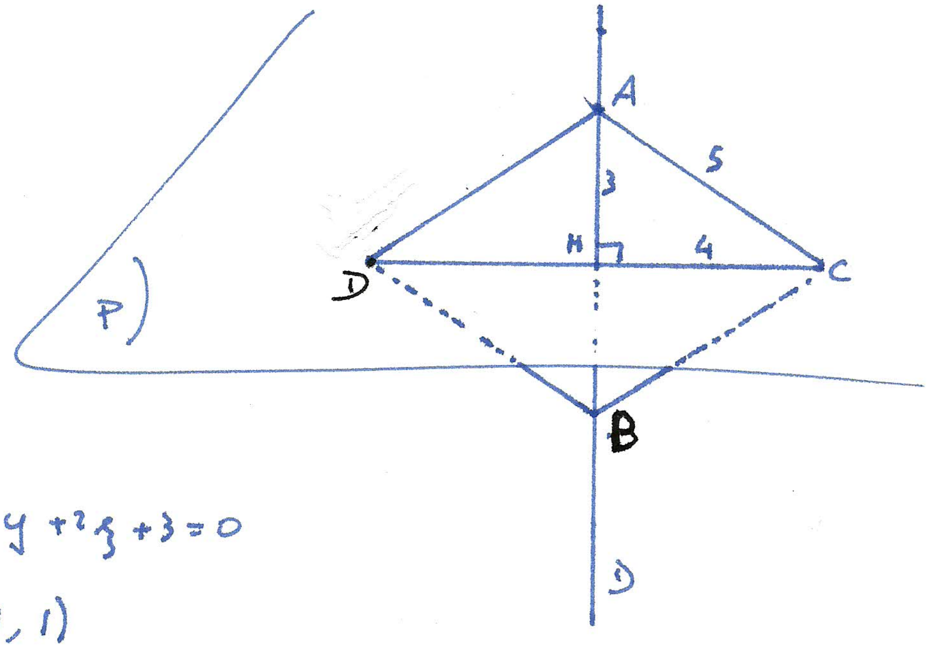
4- $|z_{n+1} - z_n| = |z_1|^n \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = 1 \times 1 = 1 = A_n A_{n+1}$.

5- $OA_n = OA_{n+1} = A_n A_{n+1} = 1 \Rightarrow OA_n A_{n+1}$ équilatéral

$\Rightarrow \widehat{A_n O A_{n+1}} = \frac{\pi}{3}$.

6- $6 \times \frac{\pi}{3} = 2\pi \Rightarrow A_0 = A_6$.

Ex 3 :



$P : x + 2y + 2z + 3 = 0$

$A(2, 1, 1)$

1- $\vec{n}\begin{pmatrix}1\\2\\2\end{pmatrix}$ donc $D : t \longmapsto \begin{cases} x = 2 + t \\ y = 1 + 2t \\ z = 1 + 2t \end{cases}$

2- $H \in P \cap D \Rightarrow 2 + t + 2(1+2t) + 2(1+2t) + 3 = 0$

$\Rightarrow 9 + 9t = -1 \Rightarrow H(1, -1, -1)$.

3- $\vec{HA}\begin{pmatrix}1\\2\\2\end{pmatrix} \Rightarrow HA = \sqrt{1+4+4} = \sqrt{9} = 3$

4- $3 = HB = \sqrt{(1+t)^2 + 2(2+2t)^2}$

$= \sqrt{9(1+t)^2} = 3|1+t| \Rightarrow |1+t| = 1 \Rightarrow t \in \{0 ; -2\}$.

Or $t = 0$ donne le point $A \Rightarrow t = -2$,

donc $B(0, -3, -3)$.

5- $M(3,y,z)\in P \Rightarrow 3+2y+2z+3=0$

$\Rightarrow y+z=-3 \Rightarrow y=-z-3$

Or $\overrightarrow{HM}\begin{pmatrix}2\\ y+1\\ z+1\end{pmatrix}$ et $HM=4 \Rightarrow 16=4+(-z-2)^2+(z+1)^2$

$\Rightarrow 12 = 2z^2+4z+2z+5 \Rightarrow 0 = 2z^2+6z-7$

$\Delta = 36+56 = 92 = 4\times 23 \Rightarrow z = \dfrac{-6\pm 2\sqrt{23}}{4} = \dfrac{-3\pm\sqrt{23}}{2}$

et $y=-z-3 = \dfrac{-3\mp\sqrt{23}}{2}$

Donc $C\left(3;\ \dfrac{-3-\sqrt{23}}{2};\ \dfrac{-3+\sqrt{23}}{2}\right)$ et $D\left(3;\ \dfrac{-3+\sqrt{23}}{2};\ \dfrac{-3-\sqrt{23}}{2}\right)$.

6- $AC=5=AD$ (Pythagore ou calcul direct).

7- (AB) est orthogonale à toute droite de P

$\Rightarrow (AB)\perp(BC)$.

8- Soit P' le plan passant par A, B et C.

$P' : 4\sqrt{23}\,x + (9-\sqrt{23})y - (9+\sqrt{23})z - 6\sqrt{23} = 0$

Or $4\sqrt{23}\cdot 3 + (9-\sqrt{23})\left(\dfrac{-3+\sqrt{23}}{2}\right) - (9+\sqrt{23})\left(\dfrac{-3-\sqrt{23}}{2}\right) - 6\sqrt{23} \neq 0$

donc $D\notin P'$, donc A, B, C et D ne sont pas coplanaires.